Closing Tues: 6.1
Closing Thur: 6.2

Unless otherwise stated, always <u>assume interest is compounded</u>.

That is, only use *simple interest* if you are doing the 6.1 HW or if the problem says “simple interest”.
All other times assume compounding!

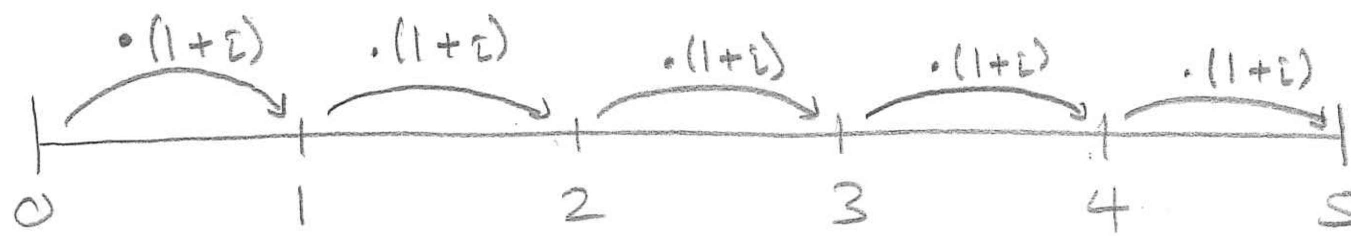
<u>6.2 Compound Interest</u>

Recall: We say interest is **compounding** if it is computed on the *entire* balance.
And we found

$$F = P(1+i)^n$$

where

$i =$ decimal interest rate
$n =$ number of compoundings



5 "compoundings"

$F = P\cdot(1+i)^5$

Bank terminology: Bank account info is given as a yearly rate divided up and used m times per year…

r = decimal interest rate,
m = "compoundings per year"
$i = r/m$ = rate used each compounding

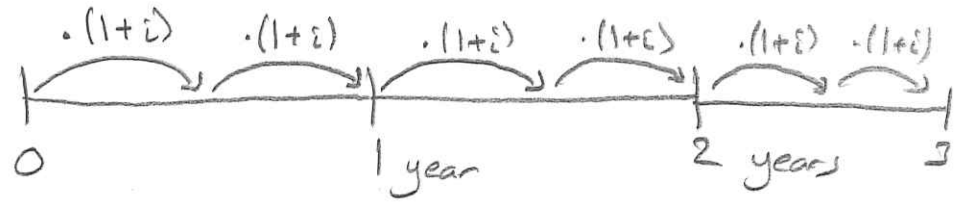
t = years
n = mt = "total compoundings"

And the formula becomes

$$F = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

Ex) 6% annual rate, compounded semi-annually (TWICE per year)

$r = 0.06$
$m = 2$
$i = \frac{r}{m} = \frac{0.06}{2} = 0.03$ } 3%, USED TWICE PER YEAR



In $t=3$ years, there will be
$n = mt = 2\cdot 3 = 6$ compoundings

$F = P\left(1+\frac{r}{m}\right)^{mt}$
$F = P\left(1+\frac{0.06}{2}\right)^{2t}$

Examples

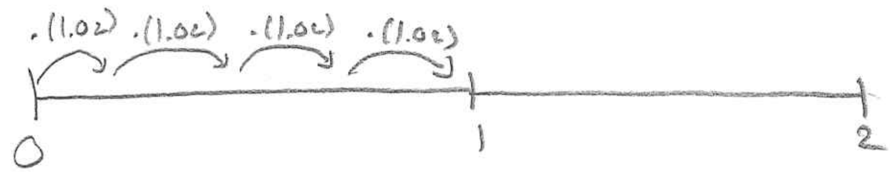
"6% per year" means *compounding* once a year (r=0.06, m=1)

$$F = P\left(1+\frac{0.06}{1}\right)^{1\cdot t} = P(1.06)^{t}$$

"6% rate, compounded semiannually" means "6/2 = 3%, twice a year" (r=0.06, *m=2*)

$$F = P\left(1+\frac{0.06}{2}\right)^{2t} = P(1.03)^{2t}$$

"8% rate, compounded quarterly" means "8/4 = 2%, 4 times a year" (r=0.08,*m=4*)



$$F = P\left(1+\frac{0.08}{4}\right)^{4t} = P(1.02)^{4t}$$

"5% rate, compounded monthly" means "5/12 = 0.4166…%, 12 times a year" (r=0.05,*m=12*)

$$F = P\left(1+\frac{0.05}{12}\right)^{12t} = P(1.0041\overline{6})^{12t}$$

Quick Examples:

(a) Harry invests $5000 in an account earning 5% per year. What is the balance in 20 years?

means compounded annually, $m=1$

$r=0.05$, $m=1$, $P=\$5000$, $t=20$, $F=?$

$$F = P\left(1+\frac{r}{m}\right)^{mt}$$

$$= 5000\left(1+\frac{0.05}{1}\right)^{(1)\cdot 20}$$

$$= 5000(1.05)^{20}$$

$$\approx 13266.4885$$

$$\boxed{\$13,266.49}$$

INTEREST $= 13266.49 - 5000$

$= \$8266.49$

more compounding, higher amount

(b) Ron invests $5000 in an account earning 5% annually, compounded quarterly. What is the balance in 20 years?

$r=0.05$, $m=4$, $P=\$5000$, $t=20$, $F=?$

$$F = P\left(1+\frac{0.05}{4}\right)^{4t}$$

$$= P(1.0125)^{4t}$$

$$F = 5000(1.0125)^{4(20)}$$

$$\approx 13507.4247$$

$$\boxed{\$13,507.42}$$

INTEREST $= 13507.42 - 5000$

$= \$8507.42$

Continuous Compounding

Assume you invest \$5000 and r = 0.03.

$$F = 5000\left(1 + \frac{0.03}{m}\right)^{m\,t}$$

compoudings	m	Balance in **one** yr.
semi-annual	2	\$5151.13
quarterly	4	\$5151.70
monthly	12	\$5152.08
daily	365	\$5152.266
hourly	87605	\$5152.27241
every minute	525600	\$5152.27267
every second	3153600	\$5152.27267

SUMMARY

$$\left.\begin{aligned} F &= P\left(1+\frac{r}{m}\right)^{mt} \\ F &= Pe^{rt} \end{aligned}\right\}$$ TWO TYPES OF COMPOUNDING ACCOUNTS

The value this is approaching is called the value from **continuous compounding**. And it is also given by

$$F = Pe^{rt}$$

ASIDE

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m = (2.71828182\ldots)^r$$

$$F = 5000e^{0.03(1)} \approx \$5152.27267$$

Quick Examples:

(a) How much must you invest at 8%, compounded monthly in order to have $10,000 in 5 years?

$r=0.08,\ m=12,\ P=?,\ F=10{,}000,\ t=5$

$$F=P\left(1+\frac{r}{m}\right)^{mt}$$

$$10000 = P\left(1+\frac{0.08}{12}\right)^{12(5)}$$

$$10000 = P\cdot(1.00\overline{6})^{60}$$

$$10000 = P\cdot 1.489845709\ldots$$

$$\Rightarrow P = \frac{10000}{1.489845709}$$

$$\approx \boxed{\$6712.\underline{10}}$$

(b) You invest $500 in an account earning 3%, compounded quarterly. How long until you have $5000?

$r=0.03,\ m=4,\ P=500,\ F=\$5000,\ t=??$

$$F=P\left(1+\frac{r}{m}\right)^{mt}$$

$$5000 = 500\left(1+\frac{0.03}{4}\right)^{4t} \quad \div 500$$

$$10 = (1.0075)^{4t}$$

$$\ln(10) = \ln\left(1.0075^{4t}\right)$$

BIG RULE! $\ln(A^B)=B\ln(A)$

$$\ln(10) = 4t\ln(1.0075)$$

$$t=\frac{\ln(10)}{4\ln(1.0075)}$$

$$\approx \boxed{77.0403 \text{ years}}$$

(c) You invest \$75 in an account where interest is computed semi-annually. After 7 years, the balance is \$210. What is the nominal rate, r?

$r = ?,\ m = 2,\ P = 75,\ F = 210,\ t = 7$

$$F = P\left(1+\frac{r}{m}\right)^{mt}$$

$$210 = 75\left(1+\frac{r}{2}\right)^{2\cdot 7}$$

$\div 75$

$$2.8 = \left(1+\frac{r}{2}\right)^{14}$$

14th root

$$\underbrace{2.8^{1/14}} = 1+\frac{r}{2}$$

$$1.076316156 \simeq 1+\frac{r}{2}$$

-1

$$0.076316156 \simeq \frac{r}{2}$$

$\cdot 2$

$$r \approx 0.152632312$$

$$\boxed{\approx 15.26\,\%}$$

(d) You place 1000 into an account that pays 4%, compounded continuously. How long does it take to double your money?

$r = 0.04,\ P = 1000,\ F = \text{DOUBLE START}$
$t = ?$

$$F = Pe^{rt}$$

$$2000 = 1000\, e^{0.04t}$$

$$2 = e^{0.04t}$$

$$\ln(2) = \ln\left(e^{0.04t}\right)$$

$$\ln(2) = 0.04t\, \underbrace{\ln(e)}_{1}$$

$$t = \frac{\ln(2)}{0.04} \approx \frac{0.69314718}{0.04}$$

$$\boxed{\approx 17.3287 \text{ years}}$$

$\boxed{\text{ASIDE}}$ $\dfrac{0.69314}{0.04} = \dfrac{69.314}{4}$

ROUGH WAY SOME FINANCE FOLKS ESTIMATE IN THEIR HEAD

"RULE OF 70 ESTIMATE"

$$\text{TIME TO DOUBLE MONEY} \simeq \frac{70}{\text{RATE}}$$

Summary of 6.2

For ALL problems in chapter 6:

1. Identify the type of account.
2. Input given facts.
3. Solve for the unknown.

Algebra Notes:

$$F = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

To solve for *P*, you just divide.
To solve for *r*, a root will be needed.
To solve for *t*, a ln( ) will be needed.

$$F = Pe^{rt}$$

To solve for *P*, you just divide.
To solve for *r*, a ln( ) will be needed.
To solve for *t*, a ln( ) will be needed.

Question:

Which is best?

A: 4%, compounded semi-annually → $APY = \left[\left(1+\frac{0.04}{2}\right)^2 - 1\right] \times 100 = 4.04$ %

B: 3.97%, compounded monthly → $APY = \left[\left(1+\frac{0.0397}{12}\right)^{12} - 1\right] \times 100 = 4.04304$ %

C: 3.955%, compounded continuously → $APY = \left[e^{0.03955} - 1\right] \times 100 = 4.03425$ %

Compute the APY for each & compare!

$\boxed{B}$ IS BEST!

$$APY = \left[\left(1+\frac{r}{m}\right)^{m(1)} - 1\right] \cdot 100\ \%$$

$$APY = \left[e^{r(1)} - 1\right] \cdot 100\ \%$$

Explaining

Annual Percentage Yield (APY) is the actual percentage change in ***one year***. We get APY by plugging in $t = 1$ year and turning the value into a percentage.

For example, for Account C, if we plug in t = 1 year we get:

$$F = Pe^{0.03955(1)} \approx P \cdot 1.0403425$$

This is the same as *multiplying* by 104.03425% which is the same as *increasing* by 4.03425%.

$(e^{0.03955} - 1) \cdot 100$

We say 4.03425% is the APY.

This is exactly what you are doing in the APY formulas.